# T-S fuzzy-affine-model-based reliable output feedback control of nonlinear systems with actuator faults*

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**Abstract**

This article presents a singular approach to the reliable $\mathscr{H}_\infty$ static output feedback (SOF) control for continuous-time nonlinear systems with Markovian jumping actuator faults. The nonlinear plants are approximated by a Takagi-Sugeno (T-S) fuzzy-affine (FA) model with parameter uncertainties, and the Markov process is adopted to characterize the actuator-fault phenomenon. Specifically, by utilizing a singular model transformation strategy, the initially constructed closed-loop system is firstly converted into a singular FA system. By constructing a Markovian Lyapunov function (MLF), together with S-procedure and some matrix inequality convexification procedures, the reliable piecewise SOF controller synthesis is then developed for the underlying systems via a convex program. Lastly, simulation examples are carried out to validate the effectiveness of the presented method.

**Keywords:** nonlinear systems, singular systems, reliable output feedback control, T-S fuzzy-affine systems, Markovian Lyapunov function.

## 1 Introduction

Recent years have witnessed the substantial studies on fuzzy control of nonlinear systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Specifically, the control scheme on the basis of the fuzzy dynamic models has aroused extensive interests due to its conceptual simplicity and rigorous effectiveness for the control of many nonlinear systems or even nonanalytic systems [13, 14, 15, 16, 17, 18, 19, 20]. Basically, the key point of T-S fuzzy models is to employ a group of fuzzy rules to characterize a complex nonlinear system by virtue of a family of local linear models, which are smoothly blended through fuzzy membership functions [21, 22]. T-S fuzzy modeling approach is essentially a multi-model protocol in which a group of linear models are integrated to characterize the global behavior of the nonlinear system. Thanks to this particular structure, researchers can fully utilize the connections between the flexible fuzzy logic theory and fruitful linear multivariable system theory, and lots of research on systematic analysis and controller design of T-S fuzzy systems has been carried out in [23, 24, 25, 26, 27, 28, 29].

Additionally, it is known that component failures commonly occur in practical engineering systems, and the existence of system component failures may lead to poor performances or even instability with the conventional feedback control protocols [30]. With the increasing requirements on system reliability, reliable control strategies have been attracting extensive concern. The aim of

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reliable control is to design a fixed controller such that the closed-loop system can sustain the stability and satisfactory performance in the context of control component failure [11, 31, 32]. Recently, some attempts have been contributed to fuzzy-model-based reliable control for nonlinear systems [10, 31, 32, 33]. For example, in [33], some results on observer-based robust and reliable control of uncertain fuzzy systems with time-varying delay were presented. In [31], the reliable $\mathscr{H}_\infty$ state-feedback controller design was proposed for nonlinear systems with actuator failures. It is noted that in [31], a single actuator-fault model with a linear function of the actuator-input was considered, which means that the life time of the system actuators will be constantly faulted or fault-free all through the operation. In [32], the problem of reliable $\mathscr{H}_\infty$ fuzzy state-feedback control was addressed for uncertain continuous-time nonlinear systems with Markov-type actuator failure. Notice that although the reliable controller problems for fuzzy dynamical systems in the form of linear subsystems have been receiving increasing concerns, little effort has been dedicated to exploiting tractable reliable SOF controller design conditions. Particularly, SOF control is of extreme significance due to the partially measurable state variables of nonlinear systems in practice. These facts render that the existing results on the SOF controller design for T-S FA systems with actuator failure leave much to be desired, which motivates us for this study.

Based on the aforementioned statements, in this article, the issue of reliable $\mathscr{H}_\infty$ SOF controller design for nonlinear systems with actuator failure will be dealt with. Especially, the nonlinear plant is characterized by an FA dynamic model with uncertainties, and the Markov process is applied to describe the actuator-fault behaviors. By employing a descriptor system approach, the closed-loop system will be firstly reformulated into a descriptor piecewise FA system. By constructing an MLF, the $\mathscr{H}_\infty$ performance analysis condition for the underlying system will be then provided, and furthermore the reliable SOF piecewise controller synthesis will be presented. It will be shown that by taking profit of the redundancy induced by a descriptor formulation, together with some convexifying procedures, the desired controller gains can be computed in a convex optimization setup. Finally, simulations will be carried out to validate the effectiveness of the presented design method. Compared with existing results on the reliable control of FA systems, the main contributions of this research are twofold: (i) A novel bounded real lemma is reformulated for FA systems in a descriptor system framework. (ii) With some matrix linearisation procedures, the reliable SOF controller synthesis for FA systems with actuator faults, parameter uncertainties and measurement noises is developed in the form of strict LMIs.

*Notations.* The notations utilized are standard. $\mathtt{Sym}\{A\}$ refers to $A + A^\top$; $\mathbb{E}[\cdot]$ refers to the mathematical expectation; signals that are square integrable over $[0, \infty)$ is represented by $L_2[0, \infty)$ with the norm $\|\cdot\|_2$.

## 2 Problem Formulation

Following the idea of [23], a general continuous-time T-S FA dynamic model can be described as the following form,

**Plant Rule $\mathscr{F}^l$: IF $f_1(x(t))$ is $F_1^l$ and $f_2(x(t))$ is $F_2^l$ and $\cdots$ and $f_g(x(t))$ is $F_g^l$, THEN**

$$\left\{\begin{array}{l} \dot{x}(t) = (A_l + \Delta A_l)x(t) + a_l + \Delta a_l + (B_{1l} + \Delta B_{1l})u(t) + D_{1l}w(t) \\ y(t) = (C_l + \Delta C_l)x(t) + D_{2l}w(t) \\ z(t) = L_l x(t) + B_{2l}u(t),\ l \in \mathscr{L} := \{1, 2, \cdots, M\} \end{array}\right. \tag{1}$$

where $\mathscr{F}^l$ denotes the $l$-th fuzzy inference rule; $M$ refers to the number of inference rules; $F_\nu^l(\nu = 1, 2, \cdots, g)$ denotes fuzzy sets; $x(t) \in \Re^{n_x}$ represents the system state; $u(t) \in \Re^{n_u}$ refers to the control input; $y(t) \in \Re^{n_y}$ represents the system measurement output; $z(t) \in \Re^{n_z}$ denotes the regulated output; $w(t) \in \Re^{n_w}$ denotes the disturbance input belonging to $L_2[0, \infty)$; $\mathbb{F}(x(t)) := [f_1(x(t)), f_2(x(t)), \cdots, f_g(x(t))]$ denote the premise variables; $(A_l, a_l, B_{1l}, B_{2l}, C_l, D_{1l}, D_{2l}, L_l)$ specifies the $l$-th local model of the system, and $(\Delta A_l, \Delta a_l, \Delta B_{1l}, \Delta C_l)$ represent the uncertainties in the

$l$-th local model, which can be normalized as

$$\begin{array}{l}\left[\begin{array}{ccc}\Delta A_l & \Delta B_{1l} & \Delta a_l\end{array}\right]=U_{1l}\Delta_l(t)\left[\begin{array}{ccc}V_{1l} & V_{2l} & V_{3l}\end{array}\right],\\ \Delta C_l=U_{2l}\Delta_l(t)V_{1l},\ l\in\mathscr{L}\end{array}\tag{2}$$

with matrices $U_{1l}$, $U_{2l}$, $V_{1l}$, $V_{2l}$, and $V_{3l}$ of appropriate dimensions. $\Delta_l(t):\ \mathbb{Z}^+\to\Re^{k_1\times k_2}$, $l\in\mathscr{L}$ are with Lebesgue measurable elements subject to

$$\Delta_l^\top(t)\Delta_l(t)\leq\mathbf{I}_{k_2},\ l\in\mathscr{L}.\tag{3}$$

Let $\mu_l[x(t)]$ be the standardized fuzzy-basis function of $F^l$ where $F^l:=\prod_{\nu=1}^{g}F_\nu^l$ and

$$\mu_l\left[x(t)\right]:=\frac{\prod_{\nu=1}^{g}\mu_{l\nu}\left[f_\nu(x(t))\right]}{\sum_{p=1}^{M}\prod_{\nu=1}^{g}\mu_{p\nu}\left[f_\nu(x(t))\right]}\geq 0,\quad\sum_{l=1}^{M}\mu_l[x(t)]=1\tag{4}$$

where $\mu_{l\nu}\left[f_\nu(x(t))\right]$ denotes the grade of membership of $f_\nu(x(t))$ in $F_\nu^l$. For simplicity, we subsequently denote $\mu_l[x(t)]$ as $\mu_l$.

By utilizing the properties of fuzzy-basis functions, the following global T-S fuzzy dynamic model can be obtained,

$$\left\{\begin{array}{l}\dot{x}(t)=(A(\mu)+\Delta A(\mu))x(t)+a(\mu)+\Delta a(\mu)+(B_1(\mu)+\Delta B_1(\mu))u(t)+D_1(\mu)w(t)\\ y(t)=(C(\mu)+\Delta C(\mu))x(t)+D_2(\mu)w(t)\\ z(t)=L(\mu)x(t)+B_2(\mu)u(t)\end{array}\right.\tag{5}$$

where

$$\chi(\mu):=\sum_{l=1}^{M}\mu_l\chi_l,\ \chi\in\{A,\Delta A,a,\Delta a,B_1,\Delta B_1,B_2,C,\Delta C,D_1,D_2,L\}.\tag{6}$$

In this paper, we tackle the reliable $\mathscr{H}_\infty$ SOF control problem for the uncertain continuous-time FA dynamic model (5) by virtue of a piecewise controller. Along the ideas in [23] and [25], we can further express the system (5) as a piecewise-FA model,

$$\left\{\begin{array}{l}\dot{x}(t)=(\mathcal{A}_i+\Delta\mathcal{A}_i)x(t)+\mathrm{a}_i+\Delta\mathrm{a}_i+(\mathcal{B}_{1i}+\Delta\mathcal{B}_{1i})u(t)+\mathcal{D}_{1i}w(t)\\ y(t)=(\mathcal{C}_i+\Delta\mathcal{C}_i)x(t)+\mathcal{D}_{2i}w(t)\\ z(t)=\mathcal{L}_ix(t)+\mathcal{B}_{2i}u(t),\ x(t)\in\mathcal{S}_i,\ i\in\mathscr{I}\end{array}\right.\tag{7}$$

where $\{\mathcal{S}_i\}_{i\in\mathscr{I}}$ and $\mathscr{I}$ refers to the state-space partition and the set of subspace indices, respectively, and

$$\left\{\begin{array}{l}\mathcal{A}_i:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sA_s,\ \Delta\mathcal{A}_i=\sum\limits_{s\in\mathscr{S}(i)}\mu_s\Delta A_s,\\ \mathrm{a}_i:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sa_s,\ \Delta\mathrm{a}_i=\sum\limits_{s\in\mathscr{S}(i)}\mu_s\Delta a_s,\\ \mathcal{B}_{1i}:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sB_{1s},\ \Delta\mathcal{B}_{1i}=\sum\limits_{s\in\mathscr{S}(i)}\mu_s\Delta B_{1s},\\ \mathcal{B}_{2i}:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sB_{2s},\ \mathcal{C}_i:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sC_s,\\ \Delta\mathcal{C}_i:=\sum\limits_{s\in\mathscr{S}(i)}\mu_s\Delta C_s,\ \mathcal{D}_{1i}:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sD_{1s},\\ \mathcal{D}_{2i}:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sD_{2s},\ \mathcal{L}_i:=\sum\limits_{s\in\mathscr{S}(i)}\mu_sL_s,\end{array}\right.\tag{8}$$

with $0<\mu_s[x(t)]\leq 1$, $\sum_{s\in\mathscr{S}(i)}\mu_s[x(t)]=1$. For each region $\mathcal{S}_i$, $i\in\mathscr{I}$, the set $\mathscr{S}(i)$ contains the indices for the subsystem modes used in the interpolation. Evidently, $\mathscr{S}(i)$ in a crisp subspace refers to only one element.

As mentioned in [23, 25], each polyhedral subspace $\mathcal{S}_i$ can be outer approximated by an ellipsoid, i.e.,

$$\mathcal{S}_i\subseteq\mathcal{E}_i,\ \mathcal{E}_i=\{x|\|Q_ix+q_i\|\leq 1\}.\tag{9}$$

Specially, if the polyhedral subspaces $\mathcal{S}_i$ are slabs with

$$\mathcal{S}_i = \{x | \varphi_{1i} \le \theta_i^\top x \le \varphi_{2i}\} \tag{10}$$

where $\varphi_{1i}, \varphi_{2i} \in \Re$, then each slab subspace can be characterized by a degenerate ellipsoid in (9) with parameters

$$Q_i = \frac{2\theta_i^\top}{\varphi_{2i} - \varphi_{1i}},\ q_i = -\frac{\varphi_{2i} + \varphi_{1i}}{\varphi_{2i} - \varphi_{1i}}. \tag{11}$$

From (9), we further attain the subsequent condition for each ellipsoid subspace,

$$\begin{bmatrix} x(t) \\ 1 \end{bmatrix}^\top \begin{bmatrix} Q_i^\top Q_i & Q_i^\top q_i \\ q_i^\top Q_i & q_i^\top q_i - 1 \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \le 0,\ i \in \mathscr{I}. \tag{12}$$

Denote the indexes of the subspaces $\mathscr{I} = \mathscr{I}_0 \cup \mathscr{I}_1$, where $\mathscr{I}_0$ involves the index set of regions with $q_i^\top q_i - 1 \le 0$ which inculdes the origin, and $\mathscr{I}_1$ represents the index set of subspaces otherwise.

For the FA system (7), we aim at designing the SOF controller as

$$u(t) = K_i y(t),\ i \in \mathscr{I} \tag{13}$$

where $K_i \in \Re^{n_u \times n_y}$, $i \in \mathscr{I}$ are the controller parameters to be designed.

For the reliable control problem formulation of the FA dynamic system in (7) with actuator faults, we characterize the actuator failure with a stochastic process,

$$u(t, r(t)) = \Sigma(r(t))u(t), \tag{14}$$

where $\Sigma(r(t)) := \mathrm{diag}\{\sigma_1(r(t)), \sigma_2(r(t)), \cdots, \sigma_{n_u}(r(t))\}$, $0 \le \sigma_\iota(r(t)) \le 1$, $\iota = 1, 2, \cdots, n_u$, and $\{r(t), t \ge 0\}$ is a Markov process describing the actuator-fault mode as described in [9].

Notice that for reliable SOF controller design, if one directly substitutes $u(t, r(t))$ in (14) into (7), then there unavoidably appear coupled terms involved by the system matrices and controller gains, i.e., $(\mathcal{B}_{1i} + \Delta\mathcal{B}_{1i})\Sigma_m K_i(\mathcal{C}_i + \Delta\mathcal{C}_i)$, where both the input and output matrices involve parametric uncertainties. This leads to that it is almost impossible to search for suitable matrices $T_{Bi}$ satisfying $T_{Bi}(\mathcal{B}_{1i} + \Delta\mathcal{B}_{1i})\Sigma_m = \begin{bmatrix} \mathbf{I}_{n_u} & \mathbf{0}_{n_u \times (n_x - n_u)} \end{bmatrix}^\top$ or matrices $T_{Ci}$ satisfying $(\mathcal{C}_i + \Delta\mathcal{C}_i)T_{Ci} = \begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times (n_x - n_y)} \end{bmatrix}$. In addition, the measured output in (1) also contains noises. Thus, the methods proposed in [34, 35] can not be utilized for SOF controller design for this case. On the other hand, due to the existence of actuator-faults in (14) it is also difficult to employ the state-input $x$-$u$ augmentation approach [36] for reliable SOF controller design. Alternatively, in this paper we may turn to a descriptor system approach with the state-output $x$-$y$ augmentation to synthesize the reliable SOF controller.

To this end, substituting $u(t, r(t))$ for $u(t)$ in (7) and taking advantage of the redundancy induced by a descriptor formulation, we obtain the following closed-loop system,

$$\begin{cases} \dot{x}(t) = (\mathcal{A}_i + \Delta\mathcal{A}_i)x(t) + \mathrm{a}_i + \Delta\mathrm{a}_i + (\mathcal{B}_{1i} + \Delta\mathcal{B}_{1i})\Sigma_m K_i y(t) + \mathcal{D}_{1i} w(t) \\ 0 \times \dot{y}(t) = (\mathcal{C}_i + \Delta\mathcal{C}_i)x(t) + \mathcal{D}_{2i} w(t), \\ z(t) = \mathcal{L}_i x(t) + \mathcal{B}_{2i}\Sigma_m K_i y(t),\ i \in \mathscr{I},\ m \in \mathcal{I} \end{cases} \tag{15}$$

which can be further expressed as

$$\begin{cases} E\dot{\bar{x}}(t) = \bar{\mathcal{A}}_{i,m}\bar{x}(t) + \bar{\mathrm{a}}_i + \bar{\mathcal{D}}_i w(t) \\ z(t) = \bar{\mathcal{C}}_{i,m}\bar{x}(t),\ i \in \mathscr{I},\ m \in \mathcal{I} \end{cases} \tag{16}$$

where $\bar{x}(t) = \begin{bmatrix} x^\top(t) & y^\top(t) \end{bmatrix}^\top$, and

$$\begin{cases} E := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \bar{\mathcal{A}}_{i,m} := \begin{bmatrix} \mathcal{A}_i + \Delta\mathcal{A}_i & (\mathcal{B}_{1i} + \Delta\mathcal{B}_{1i})\Sigma_m K_i \\ \mathcal{C}_i + \Delta\mathcal{C}_i & -\mathbf{I} \end{bmatrix}, \\ \bar{\mathcal{D}}_i := \begin{bmatrix} \mathcal{D}_{1i} \\ \mathcal{D}_{2i} \end{bmatrix},\ \bar{\mathrm{a}}_i := \begin{bmatrix} \mathrm{a}_i + \Delta\mathrm{a}_i \\ \mathbf{0} \end{bmatrix}, \\ \bar{\mathcal{C}}_{i,m} := \begin{bmatrix} \mathcal{L}_i & \mathcal{B}_{2i}\Sigma_m K_i \end{bmatrix}. \end{cases} \tag{17}$$

**Remark 2.3.** By a descriptor system representation given in (15) and (16), it is easy to see that the controller gains $K_i$ has been separated from the output matrices. This characteristic prohibits one to make any transformations to output matrices from SOF controller design purpose, which will be shown in the next section in details.

# 3 Reliable $\mathscr{H}_\infty$ SOF Controller Analysis And Design

In this section, we shall first present the bounded real lemma for the system (16), and then by some matrix inequality convexifying procedures, numerically solvable sufficient condition on the reliable and robust $\mathscr{H}_\infty$ SOF controller synthesis for the FA system (1) will be further carried out.

## 3.1 Robust $\mathscr{H}_\infty$ performance analysis

**Lemma 1.** *The closed-loop system (16) can achieve the stochastic stability with an $\mathscr{H}_\infty$ performance $\gamma$, if there exist symmetric matrices $0 < P_{m(1)} \in \Re^{n_x \times n_x}$, $m \in \mathcal{I}$, matrix $G \in \Re^{n_y \times n_y}$, and scalars $\bar{\rho}$, $\eta_i < 0$, $i \in \mathscr{I}_1$, such that the subsequent matrix inequalities hold*

$$\begin{bmatrix} -\gamma^2 \mathbf{I} & \bar{\mathcal{D}}_i^\top P_m \\ * & \Gamma_{i,m} \end{bmatrix} < 0, \ i \in \mathscr{I}_0, \tag{18}$$

$$\begin{bmatrix} -\gamma^2 \mathbf{I} & \bar{\mathcal{D}}_i^\top P_m & \mathbf{0} \\ * & \Gamma_{i,m} + \eta_i \Lambda_1^\top Q_i^\top Q_i \Lambda_1 & P_m^\top \bar{a}_i + \eta_i \Lambda_1^\top Q_i^\top q_i \\ * & * & \eta_i (q_i^\top q_i - 1) \end{bmatrix} < 0, \ i \in \mathscr{I}_1, \tag{19}$$

*where*

$$\begin{cases} \Gamma_{i,m} := \mathtt{Sym}\{\bar{\mathcal{A}}_{i,m}^\top P_m\} + \sum\limits_{n=1}^{N} \lambda_{mn} E P_n + \bar{\mathcal{C}}_{i,m}^\top \bar{\mathcal{C}}_{i,m}, \\ P_m := \begin{bmatrix} P_{m(1)} & \mathbf{0} \\ -\bar{\rho}\Lambda_2 P_{m(1)} & G \end{bmatrix}, \\ \Lambda_1 := \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} \end{bmatrix}, \ \Lambda_2 := \begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times (n_x - n_y)} \end{bmatrix}. \end{cases} \tag{20}$$

**Proof.** For the closed-loop system (16), we consider the subsequent MLF,

$$V(\bar{x}(t), r(t), t) = \bar{x}^\top(t) E P(r(t)) \bar{x}(t), \tag{21}$$

where

$$P(r(t)) = \begin{bmatrix} P_{(1)}(r(t)) & \mathbf{0} \\ -\bar{\rho}\Lambda_2 P_{(1)}(r(t)) & G \end{bmatrix} \tag{22}$$

with $0 < P_{(1)}(r(t)) \in \Re^{n_x \times n_x}$, $\Lambda_2 := \begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times (n_x - n_y)} \end{bmatrix}$, and $G \in \Re^{n_y \times n_y}$; $\bar{\rho}$ is a scalar parameter.

According to the MLF (21), the closed-loop system in (16) can achieve the stochastic stability with an $\mathscr{H}_\infty$ performance $\gamma$, if one can show that

$$\mathscr{D}[V(\bar{x}(t), r(t), t)] + \mathbb{E}\{z^\top(t) z(t)\} - \gamma^2 w^\top(t) w(t) < 0, \tag{23}$$

holds in the context of zero initial conditions for any nonzero $w(t) \in L_2[0, \infty)$, where $\mathscr{D}$ denotes the weak infinitesimal generator of the Markov process $\{\bar{x}(t), r(t), t \geq 0\}$. In light of the definition of $\mathscr{D}$ [32], we have

$$\begin{aligned} &\mathscr{D}[V(\bar{x}(t), r(t), t)] \\ &= \lim_{\delta \to 0^+} \frac{1}{\delta} \Bigg\{ \bar{x}^\top(t+\delta) E \Bigg[ \sum_{n=1}^{N} \Pr\{r(t+\delta) = n | r(t) = m\} P_n \Bigg] \bar{x}(t+\delta) \\ &\qquad - \bar{x}^\top(t) E P_m \bar{x}(t) \Bigg\} \\ &= \lim_{\delta \to 0^+} \frac{1}{\delta} \Bigg\{ \bar{x}^\top(t+\delta) E \Bigg[ \sum_{n=1}^{N} (\lambda_{mn}\delta + o(\delta)) P_n \Bigg] \bar{x}(t+\delta) \\ &\qquad + [\bar{x}(t+\delta) - \bar{x}(t)]^\top E P_m \bar{x}(t+\delta) + \bar{x}^\top(t+\delta) P_m^\top E [\bar{x}(t+\delta) - \bar{x}(t)] \Bigg\} \end{aligned}$$

$$= 2\dot{\bar{x}}^\top(t)E^\top P_m\bar{x}(t) + \bar{x}^\top(t)\left(E\sum_{n=1}^{N}\lambda_{mn}P_n\right)\bar{x}(t). \tag{24}$$

For simplicity, only the proof of condition (19) will be shown in the sequel.

Considering the system state-space equation, combined with (21), the subsequent inequality indicates (23),

$$\begin{bmatrix} w(t) \\ \bar{x}(t) \\ 1 \end{bmatrix}^\top \begin{bmatrix} -\gamma^2\mathbf{I} & \mathcal{D}_i^\top P_m & \mathbf{0} \\ * & \Gamma_{i,m} & P_m\bar{\mathrm{a}}_i \\ * & * & \mathbf{0} \end{bmatrix} \begin{bmatrix} w(t) \\ \bar{x}(t) \\ 1 \end{bmatrix} < 0, \tag{25}$$

where

$$\Gamma_{i,m} := \mathtt{Sym}\{\bar{\mathcal{A}}_{i,m}^\top P_m\} + \sum_{n=1}^{N}\lambda_{mn}EP_n + \bar{\mathcal{C}}_{i,m}^\top\bar{\mathcal{C}}_{i,m}. \tag{26}$$

Then, in view of the structural division information in (12), applying the S-procedure [23], one comes to the subsequent inequality inferring to (23) in the context of $\eta_i < 0$, $i \in \mathscr{I}_1$,

$$\mathrm{LHS}(23) + \eta_i \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^\top \begin{bmatrix} Q_i^\top Q_i & Q_i^\top q_i \\ q_i^\top Q_i & q_i^\top q_i - 1 \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \end{bmatrix} < 0, \tag{27}$$

where LHS(23) refers to the left-hand side (LHS) of inequality (23).

It is obvious that

$$x(t) = \Lambda_1\bar{x}(t),\ \Lambda_1 := \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} \end{bmatrix}. \tag{28}$$

After substituting (28) into (27), it can be readily inspected that the inequality (19) implies (23). The proof is thus completed.

In the subsequent subsection, we shall employ some convexifying techniques to solve the reliable and robust SOF controller synthesis problem.

Before developing the controller synthesis condition, we first introduce the following lemma, which will be useful for the derivation of synthesis condition.

**Lemma 2.** [37] *Let matrices $\Psi = \Psi^\top$, $\mathbb{U}$, $\mathbb{V}$, and $\Delta(t)$ be appropriate dimension. The solvability of inequality*

$$\Psi + \mathtt{Sym}\{\mathbb{U}\Delta(t)\mathbb{V}\} < 0 \tag{29}$$

*for all admissible $\Delta(t)$ satisfying $\Delta^\top(t)\Delta(t) \leq \mathbf{I}$, is of equivalence to*

$$\Psi + \epsilon\mathbb{U}\mathbb{U}^\top + \epsilon^{-1}\mathbb{V}^\top\mathbb{V} < 0 \tag{30}$$

*for some positive scalar $\epsilon$.*

### 3.2 Reliable SOF controller synthesis

**Theorem 1.** *Consider the FA system in (1). For a given scalar $\rho$, the closed-loop system (16) can achieve the stochastic stability with an $\mathscr{H}_\infty$ performance $\gamma$, if there exist symmetric matrices $0 < X_{m(1)} \in \Re^{n_x\times n_x}$, $m \in \mathcal{I}$, matrices $X_{(2)} \in \Re^{n_y\times n_y}$, $\bar{K}_i \in \Re^{n_u\times n_y}$, and scalar parameters $\epsilon_{1i} > 0$, $i \in \mathcal{I}$, $\epsilon_{2i} > 0$, and $\bar{\eta}_i < 0$, $i \in \mathcal{I}_1$, guaranteeing that the subsequent LMIs hold,*

$$\begin{bmatrix} -\epsilon_{1i}\mathbf{I} & \bar{\mathbb{V}}_{1is,m} & \mathbf{0} \\ * & \bar{\Theta}_{is,m} & \Lambda_3^\top X_{m(1)}\Pi \\ * & * & -\mathcal{X}_m \end{bmatrix} < 0,\ i \in \mathscr{I}_0, \tag{31}$$

$$\begin{bmatrix} -\epsilon_{2i}\mathbf{I} & \hat{\mathbb{V}}_{2is} & \mathbf{0} & \mathbf{0} \\ * & -\epsilon_{1i}\mathbf{I} & \hat{\mathbb{V}}_{1is,m} & \mathbf{0} \\ * & * & \hat{\Theta}_{is,m} & \hat{\Lambda}_3^\top X_{m(1)}\Pi \\ * & * & * & -\mathcal{X}_m \end{bmatrix} < 0,\ i \in \mathscr{I}_1, \tag{32}$$

*where $s \in \mathscr{S}(i)$, and*

$$\left\{\begin{array}{l}
\bar{\Theta}_{is,m} := \begin{bmatrix} -\gamma^2\mathbf{I} & \mathbf{0} & \mathscr{D}_s^\top \\ * & -\mathbf{I} & \mathbb{C}_{is,m} \\ * & * & \bar{\Gamma}_{is,m} \end{bmatrix}, \\
\bar{\Gamma}_{is,m} := \mathtt{Sym}\{\mathbb{A}_{is,m}\} + \mathrm{diag}\{\lambda_{mm}X_{m(1)} + \epsilon_{1i}U_{1s}U_{1s}^\top, \epsilon_{1i}U_{2s}U_{2s}^\top\}, \\
\mathbb{A}_{is,m} := \begin{bmatrix} A_sX_{m(1)} + B_{1s}\Sigma_m\bar{K}_i\Lambda_2 & \rho B_{1s}\Sigma_m\bar{K}_i \\ C_sX_{m(1)} - X_{(2)}\Lambda_2 & -\rho X_{(2)} \end{bmatrix}, \\
\mathbb{C}_{is,m} := \begin{bmatrix} L_sX_{m(1)} + B_{2s}\Sigma_m\bar{K}_i\Lambda_2 & \rho B_{2s}\Sigma_m\bar{K}_i \end{bmatrix}, \\
\mathscr{D}_s := \begin{bmatrix} D_{1s}^\top & D_{2s}^\top \end{bmatrix}^\top, \\
\Lambda_2 := \begin{bmatrix} \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times(n_x-n_y)} \end{bmatrix}, \\
\Lambda_3 := \begin{bmatrix} \mathbf{0}_{n_x\times(n_z+n_w)} & \mathbf{I} & \mathbf{0}_{n_x\times n_y} \end{bmatrix}, \\
\mathcal{X}_m := \mathrm{diag}\{X_{1(1)}, \cdots, X_{m-1(1)}, X_{m+1(1)}, \cdots, X_{N(1)}\}, \\
\Pi := \begin{bmatrix} \sqrt{\lambda_{m1}}\mathbf{I} & \cdots & \sqrt{\lambda_{m,m-1}}\mathbf{I} & \sqrt{\lambda_{m,m+1}}\mathbf{I} & \cdots & \sqrt{\lambda_{mN}}\mathbf{I} \end{bmatrix}, \\
\bar{\mathbb{V}}_{1is,m} := \begin{bmatrix} \mathbf{0}_{k_2\times(n_z+n_w)} & \begin{bmatrix} \bar{\Upsilon}_{is,m} & \rho V_{2s}\Sigma_m\bar{K}_i \end{bmatrix} \\ \mathbf{0}_{k_2\times(n_z+n_w)} & \begin{bmatrix} V_{1s}X_{m(1)} & \mathbf{0} \end{bmatrix} \end{bmatrix}, \\
\bar{\Upsilon}_{is,m} := V_{1s}X_{m(1)} + V_{2s}\Sigma_m\bar{K}_i\Lambda_2, \\
\hat{\Theta}_{is,m} := \begin{bmatrix} -\gamma^2\mathbf{I} & \mathbf{0} & \mathscr{D}_s^\top & \mathbf{0} \\ * & -\mathbf{I} & \mathbb{C}_{is,m} & \mathbf{0} \\ * & * & \hat{\Gamma}_{is,m} & \mathbb{X}_{is,m} \\ * & * & * & -\bar{\eta}_i(\mathbf{I} - q_iq_i^\top) \end{bmatrix}, \\
\hat{\Gamma}_{is,m} := \mathtt{Sym}\{\mathbb{A}_{is,m}\} + \mathrm{diag}\{\lambda_{mm}X_{m(1)} + \bar{\eta}_ia_sa_s^\top + (\epsilon_{1i} + \epsilon_{2i})U_{1s}U_{1s}^\top, \epsilon_{1i}U_{2s}U_{2s}^\top\}, \\
\hat{\mathbb{V}}_{1is,m} := \begin{bmatrix} \mathbf{0}_{k_2\times(n_z+n_w)} & \begin{bmatrix} \hat{\Upsilon}_{is,m} & \rho V_{2s}\Sigma_m\bar{K}_i \end{bmatrix} & \bar{\eta}_iV_{3s}q_i^\top \\ \mathbf{0}_{k_2\times(n_z+n_w)} & \begin{bmatrix} V_{1s}X_{m(1)} & \mathbf{0} \end{bmatrix} & \mathbf{0}_{k_2\times n_y} \end{bmatrix}, \\
\hat{\Upsilon}_{is,m} := V_{1s}X_{m(1)} + V_{2s}\Sigma_m\bar{K}_i\Lambda_2 + \bar{\eta}_iV_{3s}a_s^\top, \\
\hat{\mathbb{V}}_{2is} := \begin{bmatrix} \frac{1}{2}\bar{\eta}_iV_{3s}V_{3s}^\top & \mathbf{0}_{k_2\times k_2} \end{bmatrix}, \\
\hat{\Lambda}_3 := \begin{bmatrix} \mathbf{0}_{n_x\times(n_z+n_w)} & \mathbf{I} & \mathbf{0}_{n_x\times(n_y+1)} \end{bmatrix}, \\
\mathbb{X}_{is,m} := \begin{bmatrix} Q_iX_{m(1)} + \bar{\eta}_iq_ia_s^\top & \mathbf{0}_{n_y\times n_y} \end{bmatrix}^\top.
\end{array}\right. \tag{33}$$

*Specifically, the admissible SOF controller gains can be calculated as,*

$$K_i = \bar{K}_iX_{(2)}^{-1},\ i \in \mathscr{I}. \tag{34}$$

**Proof.** In view of Lemma 1, if there exist symmetric matrices $P_{m(1)} > 0$, $m \in \mathcal{I}$, matrix $G$, and scalars $\bar{\rho}$, $\eta_i < 0$, $i \in \mathscr{I}_1$ satisfying (18) and (19), then the closed-loop fuzzy system (16) can achieve stochastic stability with an $\mathscr{H}_\infty$ performance $\gamma$. Similarly, we only give the proof of the more complex condition (32).

On the basis of (8) and (17), the LHS of inequality (19) can be re-expressed as

$$\mathrm{LHS}(19) = \sum_{s\in\mathscr{S}(i)} \mu_s\Theta_{is,m},\ i \in \mathscr{I}_1,\ m \in \mathcal{I}, \tag{35}$$

where

$$\left\{\begin{array}{l}
\Theta_{is,m} := \begin{bmatrix} -\gamma^2\mathbf{I} & \mathscr{D}_s^\top P_m & \mathbf{0} \\ * & \check{\Gamma}_{is,m} + \eta_i\Lambda_1^\top Q_i^\top Q_i\Lambda_1 & P_m^\top\bar{a}_s + \eta_i\Lambda_1^\top Q_i^\top q_i \\ * & * & \eta_i(q_i^\top q_i - 1) \end{bmatrix}, \\
\check{\Gamma}_{i,m} := \mathtt{Sym}\{\mathscr{A}_{is,m}^\top P_m\} + \sum\limits_{n=1}^{N} \lambda_{mn}EP_n + \mathscr{C}_{is,m}^\top\mathscr{C}_{is,m}, \\
\mathscr{A}_{is,m} := \begin{bmatrix} A_s + \Delta A_s & (B_{1s} + \Delta B_{1s})\Sigma_mK_i \\ C_s + \Delta C_s & -\mathbf{I} \end{bmatrix}, \\
\bar{a}_s := \begin{bmatrix} a_s + \Delta a_s \\ \mathbf{0} \end{bmatrix},\ \mathscr{D}_s := \begin{bmatrix} D_{1s} \\ D_{2s} \end{bmatrix}, \\
\mathscr{C}_{is,m} := \begin{bmatrix} L_s & B_{2s}\Sigma_mK_i \end{bmatrix}.
\end{array}\right. \tag{36}$$

Owing to the fact that the fuzzy-basis functions are with intrinsically nonnegative property, then by the Schur complement, one easily attains the following inequality indicating (19)

$$\Theta_{is,m} < 0,\ i \in \mathscr{I}_1,\ s \in \mathscr{S}(i),\ m \in \mathcal{I}. \tag{37}$$

Performing a pre- and post-multiplication to (37) by $\mathrm{diag}\{\mathbf{I}_{n_w}, X_m, \mathbf{I}_1\}$ with $X_m := P_m^{-1}$ and via Schur complement, one has

$$\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathscr{C}_{is,m}X_m & \mathbf{0} \\ * & -\gamma^2\mathbf{I} & \mathscr{D}_s^\top & \mathbf{0} \\ * & * & \breve{\Gamma}_{is,m} + \eta_i X_m \Lambda_1^\top Q_i^\top Q_i \Lambda_1 X_m & \bar{a}_s + \eta_i X_m \Lambda_1^\top Q_i^\top q_i \\ * & * & * & \eta_i(q_i^\top q_i - 1) \end{bmatrix} < 0, \tag{38}$$

where

$$\breve{\Gamma}_{is,m} := \mathtt{Sym}\{\mathscr{A}_{is,m}X_m\} + \sum_{n=1}^{N} \lambda_{mn} E X_m X_n^{-1} X_m. \tag{39}$$

Additionally, considering the terms $\eta_i X_m \Lambda_1^\top Q_i^\top Q_i \Lambda_1 X_m$ in (38), the Lyapunov matrices $X_m$ are coupled with the terms $\eta_i$. To realize the controller synthesis, it is necessary to divide the term $\eta_i$ from the Lyapunov matrices $X_m$. Specially, by Schur complement, (38) can be equivalently rewritten as

$$\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathscr{C}_{is,m}X_m \\ * & -\gamma^2\mathbf{I} & \mathscr{D}_s^\top \\ * & * & \breve{\Gamma}_{is,m} + \eta_i X_m \Lambda_1^\top Q_i^\top Q_i \Lambda_1 X_m \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \bar{a}_s + \eta_i X_m \Lambda_1^\top Q_i^\top q_i \end{bmatrix} \eta_i^{-1}(q_i^\top q_i - 1)^{-1}(*) < 0. \tag{40}$$

Recalling the Matrix Inverse Lemma

$$(T + UWV)^{-1} = T^{-1} - T^{-1}U(W^{-1} + VT^{-1}U)^{-1}VT^{-1}, \tag{41}$$

we have

$$\begin{cases} \eta_i^{-1}\bar{a}_s(1 - q_i^\top q_i)^{-1}\bar{a}_s^\top = \eta_i^{-1}\bar{a}_s\bar{a}_s^\top + \eta_i^{-1}\bar{a}_s q_i^\top(\mathbf{I} - q_i q_i^\top)^{-1} q_i \bar{a}_s^\top, \\ \bar{a}_s(1 - q_i^\top q_i)^{-1} q_i^\top Q_i \Lambda_1 X_m = \bar{a}_s q_i^\top (\mathbf{I} - q_i q_i^\top)^{-1} Q_i \Lambda_1 X_m, \\ \eta_i X_m \Lambda_1^\top Q_i^\top q_i (1 - q_i^\top q_i)^{-1} q_i^\top Q_i \Lambda_1 X_m \\ = \eta_i X_m \Lambda_1^\top Q_i^\top (\mathbf{I} - q_i q_i^\top)^{-1} Q_i \Lambda_1 X_m - \eta_i X_m \Lambda_1^\top Q_i^\top Q_i \Lambda_1 X_m. \end{cases} \tag{42}$$

Therefore, in accordance with the equalities shown in (42), the inequality (40) can be transformed below in the context of $i \in \mathscr{I}_1$

$$\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathscr{C}_{is,m}X_m & \mathbf{0} \\ * & -\gamma^2\mathbf{I} & \mathscr{D}_s^\top & \mathbf{0} \\ * & * & \breve{\Gamma}_{is,m} + \eta_i^{-1}\bar{a}_s\bar{a}_s^\top & X_m \Lambda_1^\top Q_i^\top + \eta_i^{-1}\bar{a}_s q_i^\top \\ * & * & * & -\eta_i^{-1}(\mathbf{I} - q_i q_i^\top) \end{bmatrix} < 0. \tag{43}$$

Define $X_m := P_m^{-1}$, $X_{m(1)} := P_{m(1)}^{-1}$, $X_{(2)} := P_{(2)}^{-1}$, $\rho := \bar{\rho}^{-1}$. Then, we have

$$X_m = \begin{bmatrix} X_{m(1)} & \mathbf{0} \\ X_{(2)}\Lambda_2 & \rho X_{(2)} \end{bmatrix}. \tag{44}$$

Furthermore, we compute,

$$\begin{cases} \sum_{n=1}^{N} \lambda_{mn} X_m^\top E X_n^{-1} X_m = \operatorname{diag}\Big\{ \sum_{n=1}^{N} \lambda_{mn} X_{m(1)}^\top X_{n(1)}^{-1} X_{m(1)}, \mathbf{0} \Big\}, \\ \begin{bmatrix} \bar{A}_s & \bar{B}_{1s}\Sigma_m K_i \\ \bar{C}_s & -\mathbf{I} \end{bmatrix} \begin{bmatrix} X_{m(1)} & \mathbf{0} \\ X_{(2)}\Lambda_2 & \rho X_{(2)} \end{bmatrix} := \begin{bmatrix} \bar{A}_s X_{m(1)} + \bar{B}_{1s}\Sigma_m \bar{K}_i \Lambda_2 & \rho \bar{B}_{1s}\Sigma_m \bar{K}_i \\ \bar{C}_s X_{m(1)} - X_{(2)}\Lambda_2 & -\rho X_{(2)} \end{bmatrix}, \\ \begin{bmatrix} L_s & B_{2s}\Sigma_m K_i \end{bmatrix} \begin{bmatrix} X_{m(1)} & \mathbf{0} \\ X_{(2)}\Lambda_2 & \rho X_{(2)} \end{bmatrix} := \begin{bmatrix} L_s X_{m(1)} + B_{2s}\Sigma_m \bar{K}_i \Lambda_2 & \rho B_{2s}\Sigma_m \bar{K}_i \end{bmatrix}, \\ Q_i \Lambda_1 X_m = \begin{bmatrix} Q_i X_{m(1)} & \mathbf{0} \end{bmatrix}, \ \bar{K}_i := K_i X_{(2)}, \\ \bar{A}_s := A_s + \Delta A_s, \ \bar{B}_{1s} := B_{1s} + \Delta B_{1s}, \ \bar{C}_s := C_s + \Delta C_s. \end{cases} \tag{45}$$

Now, substituting the matrices defined in (45) into (43), and by Schur complement, together with the consideration of the parametric uncertainties shown in (2)-(3), one can readily re-organize the LHS of inequality (43) as

$$\mathrm{LHS}(43) = \begin{bmatrix} \tilde{\Theta}_{is,m} + \mathtt{Sym}\{\tilde{\mathbb{U}}_s \Delta_s(t) \tilde{\mathbb{V}}_{is,m}\} & \hat{\Lambda}_3^\top X_{m(1)} \Pi \\ * & -\mathcal{X}_m \end{bmatrix}, \tag{46}$$

where

$$\begin{cases} \tilde{\mathbb{U}}_s := \begin{bmatrix} \mathbf{0}_{k_1 \times (n_z+n_w)} & \begin{bmatrix} U_{1s}^\top & \mathbf{0}_{k_1 \times n_y} \end{bmatrix} & \mathbf{0}_{k_1 \times n_y} \\ \mathbf{0}_{k_1 \times (n_z+n_w)} & \begin{bmatrix} \mathbf{0}_{k_1 \times n_x} & U_{2s}^\top \end{bmatrix} & \mathbf{0}_{k_1 \times n_y} \end{bmatrix}, \\ \tilde{\mathbb{V}}_{is,m} := \begin{bmatrix} \mathbf{0}_{k_2 \times (n_z+n_w)} & \begin{bmatrix} \tilde{\Upsilon}_{is,m} & \rho V_{2s}\Sigma_m \bar{K}_i \end{bmatrix} & \bar{\eta}_i V_{3s} q_i^\top \\ \mathbf{0}_{k_2 \times (n_z+n_w)} & \begin{bmatrix} V_{1s} X_{m(1)} & \mathbf{0} \end{bmatrix} & \mathbf{0}_{k_2 \times n_y} \end{bmatrix}, \\ \tilde{\Upsilon}_{is,m} := V_{1s} X_{m(1)} + V_{2s}\Sigma_m \bar{K}_i \Lambda_2 + \bar{\eta}_i V_{3s} a_s^\top + \frac{1}{2}\bar{\eta}_i V_{3s} V_{3s}^\top \Delta_s^\top(t) U_{1s}^\top, \\ \tilde{\Theta}_{is,m} := \begin{bmatrix} -\gamma^2 \mathbf{I} & \mathbf{0} & \mathscr{D}_s^\top & \mathbf{0} \\ * & -\mathbf{I} & \mathbb{C}_{is,m} & \mathbf{0} \\ * & * & \tilde{\Gamma}_{is,m} & \mathbb{X}_{is,m} \\ * & * & * & -\bar{\eta}_i(\mathbf{I} - q_i q_i^\top) \end{bmatrix}, \\ \tilde{\Gamma}_{is,m} := \mathtt{Sym}\{\mathbb{A}_{is,m}\} + \operatorname{diag}\{\lambda_{mm} X_{m(1)}, \mathbf{0}\}, \\ \bar{\eta}_i := \eta_i^{-1}, \end{cases} \tag{47}$$

and matrices $\mathbb{A}_{is,m}$, $\mathbb{C}_{is,m}$, $\mathscr{D}_s$, $\mathbb{X}_{is,m}$, $\Lambda_2$, $\hat{\Lambda}_3$, $\Pi$, and $\mathcal{X}_m$ are defined in (33).

Hence, introducing two groups of scalar parameters $\epsilon_{1i} > 0$ and $\epsilon_{2i} > 0$, $i \in \mathscr{I}_1$, and the application of Lemma 2 to eliminate the parametric uncertainties in (46), it is easy to conclude that (32) implies (43).

On the other hand, $\mathtt{Sym}\{-X_{(2)}\} < 0$ is indicated by the condition (32), which further means that $X_{(2)}$ is nonsingular. Therefore, the controller gains can be calculated by (34). This completes the proof.

**Remark 3.1.** Theorem 1 presents a reliable $\mathscr{H}_\infty$ SOF controller synthesis condition for FA systems with Markov-type actuator failure. Notice that no transformations are performed to the input/output matrices and no structural constraints are pushed on the mode-dependent Markovian Lyapunov matrices $X_{m(1)}$, $m \in \mathcal{I}$, which show the advantages of the singular system-based approach.

# 4 Simulation Studies

**Example** Consider a T-S FA dynamic system in (1), which is composed of three local models. The system matrices are followed with

$$\left[\begin{array}{c|c|c|c} A_1 & B_{11} & D_{11} & a_1 \\ \hline C_1 & & D_{21} & \\ \hline L_1 & B_{21} & & \end{array}\right] = \left[\begin{array}{cc|c|c|c} 1.2 & 3 & 1.5 & 0 & 0 \\ 1 & -2 & 0.02 & 0.5 & 0 \\ \hline 1 & 0 & & 0.5 & \\ \hline 0 & 1 & 0.5 & & \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} A_2 & B_{12} & D_{12} & a_2 \\ \hline C_2 & & D_{22} & \\ \hline L_2 & B_{22} & & \end{array}\right] = \left[\begin{array}{cc|c|c|c} 1 & -1.3 & 1.5 & 0 & 0 \\ -0.1 & -0.7 & 0.02 & 0.5 & 0.6 \\ \hline 1 & 0 & & 0.5 & \\ \hline 0 & 1 & 0.5 & & \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} A_3 & B_{13} & D_{13} & a_3 \\ \hline C_3 & & D_{23} & \\ \hline L_3 & B_{23} & & \end{array}\right] = \left[\begin{array}{cc|c|c|c} 1 & 2 & 1.5 & 0 & 0 \\ -1 & -0.1 & 0.02 & 0.5 & -0.6 \\ \hline 1 & 0 & & 0.5 & \\ \hline 0 & 1 & 0.5 & & \end{array}\right],$$

and the parametric uncertainties are given by (2) with

$$\begin{array}{l} \left[\begin{array}{c|c|c|c|c|c} U_{1l}^{\top} & U_{2l} & V_{1l} & V_{2l} & V_{3l} & V_{4l} \end{array}\right] \\ = \left[\begin{array}{cc|c|cc|cc|c|c} 0 & 0.1 & 0.1 & 0.2 & 0 & 0.1 & 0.1 & 0.1 & 0.2 \end{array}\right], \ l = 1, 2, 3. \end{array}$$

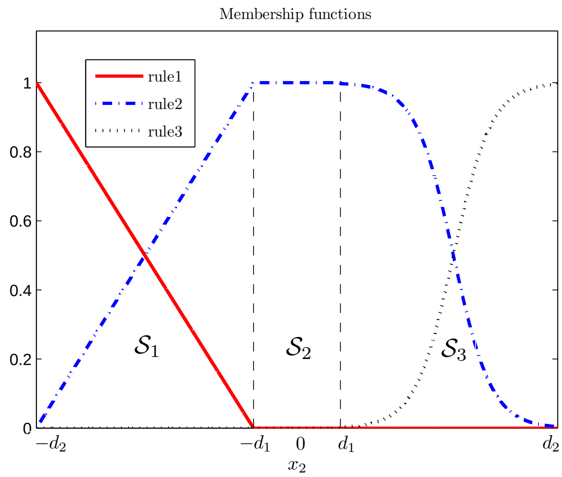


Figure 1: Membership functions

The membership functions are depicted in Figure 1. Considering the space partition that is defined in the previous section, we attain the following three subspaces $\mathcal{S}_1 := \{x \in \Re^2 | -d_2 \le x_2 \le -d_1\}$, $\mathcal{S}_2 := \{x \in \Re^2 | -d_1 \le x_2 \le d_1\}$, $\mathcal{S}_3 := \{x \in \Re^2 | d_1 \le x_2 \le d_2\}$, where $d_1$ and $d_2$ are selected to be $d_1 = 5$ and $d_2 = 30$, respectively. Here the three ellipsoid subspaces can be exactly characterized by

$$\mathcal{E}_i := \{x \in \Re^2 | \|Q_i x + q_i\| \le 1\}, \ i = 1, 2, 3$$

with

$$\begin{array}{l} Q_1 = \left[\begin{array}{cc} \frac{2}{d_2 - d_1} & 0 \end{array}\right], \ q_1 = \frac{d_2 + d_1}{d_2 - d_1}, \\ Q_2 = \left[\begin{array}{cc} \frac{1}{d_1} & 0 \end{array}\right], \ q_2 = 0, \\ Q_3 = \left[\begin{array}{cc} \frac{2}{d_2 - d_1} & 0 \end{array}\right], \ q_3 = \frac{d_2 + d_1}{d_1 - d_2}. \end{array}$$

From the subspaces shown in Figure 1, we have $\mathscr{I} = \{1, 2, 3\}$, $\mathscr{S}(1) = \{1, 2\}$, $\mathscr{S}(2) = \{1\}$, $\mathscr{S}(3) = \{1, 3\}$. It can be easily inspected that $\mathcal{S}_2$ is a crisp subspace, and $\mathcal{S}_1$ and $\mathcal{S}_3$ are fuzzy subspaces. Assume that there may exist actuator faults with 50% reduction in signal strength. Then, we obtain that $\mathcal{I} = \{1, 2\}$, $\Sigma_1 = 1$, and $\Sigma_2 = 0.5$. The actuator fault rates are supposed to be $\lambda_{11} = -0.5$, $\lambda_{12} = 0.5$, $\lambda_{21} = 10$, and $\lambda_{22} = -10$.

The purpose hereof is to synthesize a reliable piecewise controller (13) guaranteeing the stochastic stability of the corresponding closed-loop system (16), and with an $\mathscr{H}_\infty$ performance $\gamma$ under actuator faults. *It is found that the schemes developed in [22, 34, 35] are not applicable for the controller design of this class of FA systems.* However, resorting to Theorem 1 with $\rho = 0.3$, leads to desired solutions for the reliable controllers with $\mathscr{H}_\infty$ performance index $\gamma_{\min} = 2.1057$, and the controller parameters are computed as,

$$\left[\begin{array}{c|c|c} K_1 & K_2 & K_3 \end{array}\right] = \left[\begin{array}{c|c|c} -5.4026 & -5.2392 & -5.3855 \end{array}\right].$$

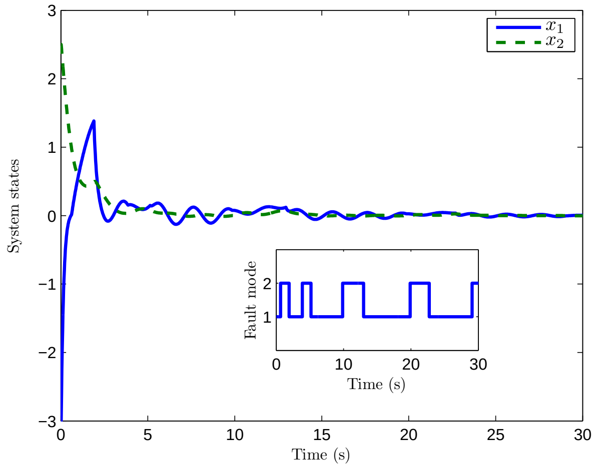


Figure 2: State trajectories of the closed-loop fuzzy system

To show the effectiveness of the obtained results, simulations will be performed with some initial conditions. Specifically, let the initial condition $x(0) = \left[\begin{array}{cc} -3 & 2.5 \end{array}\right]^\top$, and choose the disturbance input $w(t) = 2\mathrm{e}^{-0.01t}\sin(\pi t)$, $\Delta_l(t) = \sin(\pi t)$, $l = 1, 2, 3$. Figure 2 shows the state responses of the closed-loop system under the actuator fault transition information $r(t)$, and Figure 3 depicts the time response of the ratio $\sqrt{\int_0^{T_f} z^\top(t)z(t)\mathrm{d}t}/\sqrt{\int_0^{T_f} w^\top(t)w(t)\mathrm{d}t}$, which is obviously less than the calculated minimal performance index 2.1057. It is obvious from the state trajectories that, despite the actuator-faults existing in the operation of the underlying system, the synthesized reliable SOF controller can stabilize the above fuzzy system with satisfactory performance.

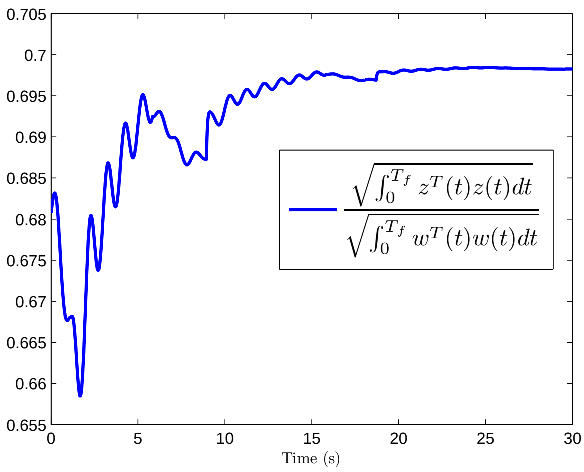


Figure 3: Response of the ratio $\sqrt{\int_0^{T_f} z^\top(t)z(t)\mathrm{d}t}/\sqrt{\int_0^{T_f} w^\top(t)w(t)\mathrm{d}t}$

## 5 Conclusions

This paper has addressed the problem of reliable $\mathscr{H}_\infty$ SOF control for nonlinear systems with actuator faults. The T-S FA model has been applied to approximate the nonlinear plant, and the Markov process has been employed to characterize the actuator-fault behaviors. In particular, by utilizing a system augmentation approach, the initial constructed closed-loop system has been firstly reformulated into the singular FA system. On the basis of an MLF, the robust performance analysis condition for the underlying singular system has been then derived, and furthermore the reliable SOF controller synthesis has been presented. It has been shown that by invoking the redundancy properties induced by the descriptor formulation, together with some convexifying techniques, the desired reliable controller can be attained. Lastly, simulation examples have verified the effectiveness of the proposed design method.

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